A Method for Detecting Possible Non-determinism in a Time Series

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A method for detecting possible non-deterministic dynamics underlying a time series is introduced. Non-deterministic dynamics may arise due to the failure of the Lipschitz condition in the equations of motion. At a singular point, the phase space trajectory of the system may jump from one solution to another. This discontinous change implies divergence of the second derivative of the solution whenever it passes near the singular point. A time series can be examined for such divergences, which may indicate non-determinism in the dynamical system. Examples with both simulated and actual data are given.

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a. Non-deterministic dynamics The standard approach to classical dynamics assumes a Laplacian point of view, i.e., that the time evolution of a system is uniquely determined by it's initial conditions [1]. This view of determinism results from the *Existence and Uniqueness Theorem* of differential equations [2], which requires that the equations of motion everywhere satisfy the Lipschitz condition. It has long been tacitly assumed that nature (in the classical realm) is deterministic, and that correspondingly, the equations of motion describing physical systems are Lipschitz.

However, there is no *a priori* reason to believe that nature is Lipschitzian. For instance, it has been shown [3] that in the case of a whip snapping, the physical solutions correspond to equations of motion that violate the Lipschitz condition. A similar effect is seen when seismic waves approach the surface of the earth [4]. In this paper, we are concerned with a particular implication of non-Lipschitz equations of motion, namely, the possibility of non-unique solutions. If a dynamical system is non-Lipschitz at a singular point, it is possible that several solutions will intersect at this point [5,6]. Since this singularity is a common point among many trajectories, the dynamics of the system *after* the singular point is intersected is not in any way determined by the dynamics before, hence the term *non-deterministic dynamics.* We hasten to emphasize, that the term "non-deterministic" should not necessarily be construed to mean purely random. On the contrary, careful analysis demonstrates unique, and complicated dynamics [5–7], some of which we describe here.

Of consequence to us in this paper is in possibility of *non-deterministic chaos.* For a non-deterministic system, it is entirely possible (if not likely), that as the various solutions move away from the singularity they will evolve very differently, and tend to diverge. Several solutions coincide at the non-Lipschitz singularity, and therefore whenever a phase space trajectory comes near this point any arbitrarily small perturbation may push the trajectory on to a completely different solution. As "noise" is intrinsic to any physical system, we expect the time evolution of a non-deterministic dynamical system to consist of a series of transient trajectories, with a new one being chosen randomly whenever the solution (in the presence of noise) nears the non-Lipschitz point. We term such behavior non-deterministic chaos.

b. Example of non-deterministic chaos A physically motivated non-deterministic set of equations comes from a simple model describing the dynamics of neutron star magnetic fields [8]. The model envisions a neutron star

to contain two spherical components, each carrying opposite charge. These charged spheres are allowed to rotate differentially, and interact both mechanically and electromagnetically. The mechanical interaction is a simple damping, taken to be proportional to the difference in the two angular velocities. The electromagnetic interactions are a standard gyromagnetic term, which induces precession of the magnetic moments, and the Landau-Lifshitz magnetic damping [9], which tends to align the magnetic moments.

When appropriately scaled and expressed in cyclindrical polar coordinates, the equations of motion of the magnetic moment due to the rotation of one sphere take the form

$$\dot{\rho} = \frac{\rho z}{\rho^2 + z^2} - \epsilon\rho,$$
$$\dot{z} = \frac{z^2}{\rho^2 + z^2} - \epsilon z + \epsilon - 1, \tag{1}$$

where ρ and z are the scaled radial and z components of the magnetic moment, and ϵ is a parameter describing the relative strengths of the mechanical and Landau-Lifshitz damping. The solutions of eqs. 1 are a series of closed loops, all sharing a common tangent point at the origin [7]. In the presence of external fluctuations, eqs. 1 should exhibit non-deterministic chaos. A numerically simulated time series for eqs. 1 is shown in Fig. 1.

For the above example, the non-Lipschitz singularity occurs at a non-equilibrium point. Other examples where the non-Lipschitz behavior is seen at an equilibrium point have been previously given [5]. In such cases, time variation of the stability of the equilibrium point may lead to non-deterministic chaos [5,10].

c. Detecting non-determinism in time series   Suppose for a moment that the time series of Fig. 1 is a data set, for which we know nothing of the underlying dynamical system. Several approaches have been proposed to detect determinism in an otherwise random appearing data set. However, such approaches *assume* that nature is continuously differentiable, which is clearly not necessarily the case. It has been shown [11] that such assumptions may lead one to believe that the system is more complex than in actually is.

It is thus important to be able to detect non-determinism in data, so that the *apparent* complexity is not mistaken for actual complexity in the underlying physical system. The key feature of non-deterministic chaos is the abrupt change of the trajectory as it jumps from one solution to another every time it nears the singularity. Treating this jump as essentially instantaneous, we might write the solution

$$x(t) = \Theta(t - t_0)x_1(t) + \Theta(t_0 - t)x_2(t), \tag{2}$$

where $\Theta(x)$ is the unit step function, and t_0 is the time when the jump occurs. If $x_1(t)$ and $x_2(t)$ are non-trivial and different solutions, then some time derivative of $x(t)$ will diverge at t_0. Indeed, analysis of the non-Lipschitz nature of the equations of motion indicate that the divergence would be seen in the second (or higher) time derivative.

d. Numerical example   We take our numerical example from the data set shown in Fig. 1. This time series was generated by 4th order Runge-Kutta integration of eqs. 1 with a step size of 0.0001. At each integration step, a Gaussian random number with zero mean and standard deviation of 10^{-8} was added to the dependent variables. To mimic the effect of finite sampling, only every 100th data point was kept. The second time derivative of $z(t)$ was computed numerically using Lagrangian interpolation, and is shown in Fig. 2. The divergences are plainly visible, corresponding to positive slope zero crossings of the signal. Such divergences are indicative of a non-deterministic jump in the trajectory, and indeed, study of eqs. 1 indicate that the singular behavior is expected at this point [7].

e. Application to physiologic data   It has been previously suggested [12] that non-determinism may appear in biological systems. We believe that such dynamics may be especially important in the case of biologic data which have a close affinity with physical models such as oscillations. Although such models are especially informative, questions of control are not easily resolved. We suspect non-deterministic chaos may prove to be an important additional feature in such models. In particular, as an infinite number of solutions are accesible by an arbitrarily small perturbation in the neighborhood of the singularity, selection of a particular solution by a control mechanism is very easy. The result would be a series of nearly identical oscillations, where slight differences in successive oscillations are randomly noise induced. Again, the derivative test may be able to detect this "controlled" non-deterministic chaos.

We have obtained the digitized records(500 Hz) of the oscillatory motion of a human forearm swinging at the elbow extended out from the shoulder. Instructions were given to the subject to maintain the oscillations at a frequency maintained by a metronome at approximately 1 Hz. Note that high frequency noise will cause large fluctuations in the time derivatives, and is unavoidable due to the physiologic nature of the system. Because the usual requirements of stationarity for signal averaging could not be assumed, the data were smoothed by boxcar averaging with a 5 point wide window applied 100 successive times. We note that this procedure is not optimal in that any filtering procedure will "smooth out" any divergence of higher order derivatives. This is equivalent to the well-known problems associated with the impulse response functions of digital filters. Clearly, sharp divergences will show up as wave functions of varying magnitude and phase dependent upon local conditions. Indeed, we suspect one reason why such divergences have not been routinely encountered has been due to the ubiquitous need to filter experimental data. Once the signal has been smoothed, Lagrangian interpolation is used to calculate the second derivative, shown in Fig. 3. The results, while far from conclusive, are suggestive. The "divergences", while not the largest oscillations, do bisect the original oscillations quite nicely, which might be expected given previous modeling results [13,14]. Further, the "divergences" show the correct sign, and while other large oscillations in the second derivative are generally associated with visible departures from smoothness in the original signal, the signal around the extrema is fairly well behaved, and thus the features in the second derivative associated with the extrema of the original signal may indicate non-deterministic behavior at these points.

Again, given the complications involved in examining this data set, we caution against any conclusive interpretation. However the results seem compelling enough to warrant further investigation.

f. Conclusions   Non-determinism may occur in a dynamical system that does not satisfy the Lipshitz condition at a point or set of points. In the presence of arbitrarily small fluctuations, this non-determinism may give rise to behavior that appears very complex. We have suggested that, given the expected piecewise continuous nature of non-deterministic chaotic signals, one possible method of detecting this phenomenon in time series is to look for divergences in some time derivative. Application to numerically simulated data has met with marked success. The derivative test, however, is limited by noise and other factors, and will be most successful on "clean" data. Appropriate smoothing and/or signal averaging may mitigate these problems.

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FIG. 1. An example of non-deterministic chaos, showing y vs. t for the numerically integrated neutron star equations in the presence of noise.

FIG. 2. A plot of the time series of Fig. 1, along with it's numerically calculated second time derivative. The plot of the second derivative has been scaled and shifted upward for clarity. Note the divergences in the second derivative which occur at positive slope zero crossings, corresponding to the location of the singular point in the neutron star model.

FIG. 3. A portion of the unsmoothed arm data (dashed curve) and the numerical second derivative of the smoothed data (solid curve). The vertical dotted lines correspond to the expected location of singular points in a non-deterministic model of human arm motion, showing a clear correspondence with peaks in the second derivative which are consistent with the expected behavior of a smoothed delta function.